Perturbative Analysis of Bianchi IX using Ashtekar Formalism

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The goal of this paper is to provide a new analysis of the classical dynamics of Bianchi type I, II and IX models by applying conventional Hamiltonian methods in the language of Ashtekhar variables. We show that Bianchi type II models can be seen as a perturbation of Bianchi I ones, and integrated. Bianchi IX models can be seen, in turn, as a perturbation of Bianchi IIs, but here the integration algorithm breaks down. This is an "interesting failure", bringing light onto the chaotic nature of Bianchi type IX dynamics.As a by product of our analysis we filled some gaps in the literature, such us recovering the BKL map in this context.

I. INTRODUCTION

The goal of this paper is to provide a new analysis of the classical dynamics of Bianchi type I, II and IX models by applying conventional Hamiltonian methods in the language of Ashtekhar variables. We show that Bianchi type II models can be seen as a perturbation of Bianchi I ones, and integrated. Bianchi IX models can be seen, in turn, as a perturbation of Bianchi IIs, but here the integration algorithm breaks down. This is an "interesting failure", bringing light onto the chaotic nature of Bianchi type IX dynamics.

Bianchi models can be traced back to the nineteen century [1], at least in the abstract, but their use in physical considerations had of course to wait for the invention of General Relativity (GR) and the development of cosmology. Bianchi I was first discussed by Kasner [2] and further by Taub [3], Misner [5] and Lifshitz and Khalatnikov [4].The empty Bianchi II model was solved first by Taub in a work where he discussed empty spaces with a three parameter group of motion [3]. Misner reformulated homogeneous cosmology using Hamiltonian methods [6], after the general hamiltonian formulation due to Arnowitt,

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Deser and Misner (ADM) [7]. This led Misner to introduce the fruitful concept of minisuperspace [8] where the cosmological evolution was equivalent to the motion of a particle in a potential. The cosmological implications of Bianchi IX were treated by Misner [9] and by Belinski, Lifshitz and Khalatnikov (BKL) [10]. We should note that when we write Bianchi IX, we restrict ourselves to the empty, diagonal case. [11]

Bianchi IX is a homogeneous solution that became popular as a generic model of the approach to the singularity, even in inhomogeneous models [4]. Although this is a controversial point [12], the approach of treating the inhomogeneous case near the singularity as a perturbation (where the inhomogeneity were taking un account by a development in spatial derivatives) from a zero order homogeneous solution (Bianchi IX) has been fruitfull [13]. In any case, the approach to the singularity as viewed by BKL shed new light on the dynamics of this complex solution since it can for be appproximated as a one dimensional map. During most of its evolution, Bianchi IX can be seen as a sequence of Kasner solutions, with the parameters of one Kasner "era" being mapped into those of the next; Barrow proved this map to be chaotic [14].

Actually, this was not the first time that chaos was proven in an approximation of Bianchi IX. Chitre [15] proved that the Misner approximation of Bianchi IX was chaotic analitically using a know theorems about geodesic flows and the hyperbolic plane (see the article of Misner [16] for a summary). Zardecki [17] first made numerical integrations of the Einstein Equations describing the Binchi IX evolution in the BKL form and obtained results that seemed to agree with these analysis. However, further investigations by Francisco and Matsas [18] and Rugh [19] showed that the initial conditions posed by Zardecki did not obey the scalar constraint. This could introduce negative energy densities, preventing collapse at the singularity and setting up chaotic motions. Furthermore, Francisco and Matsas computed the first Lyapunov exponent and showed that it tended toward zero. Analytical results from Burd et al. [20] and Hobill [21] showed that the Lyapunov exponents had to be identically zero. Pullin and Rugh [22] [23] explained this result by proving that the use of different time parametrizations could produce different Lyapunov exponents. The situation became confused, as usual chaos indicators seemed unable to capture the "irregular " character of (deterministic) Bianchi IX. By example a first analysis using The Painlevé test seemed to indicate that it was integrable [24] but this claim was contested shortly thereafter [25] [26] using the perturbative Painlevé test. Again, these works strongly indicated that the Mixmaster universe is indeed chaotic but we ought to remember that neither the Painlevé test nor the perturbative one have reached theorem status, that is, at this time, they can only be taken as chaotic behavior indicators. The

latest original indication toward the chaotic character of the Mixmaster was given by Cornish and Levin [27]. They uncovered a (multi-)fractal repellor both in a two parameter map, the so-called Farey map [28], and in the full continuous dynamics. Fractality could be good chaos indicator in the context of GR since they cannot be undone via diffeomorphism. In the exact case however, the fractal structure is obtained numerically; as experience dictates, this is a tool to be use with caution with Bianchi IX.

The main technical tool of our analysis is the appeal to Ashtekar's variables, introduced in the paper of Ashtekar of 1986 [29]. Ashtekars motivation was the quantization of GR, but he himself worked in the application of this new formalism in Bianchi cosmologies [30]. This was not the first time however that Ashtekar's formalism was applied to Bianchi cosmology [31]. The introduction of the tetrad and first order formalisms was achieved in [32]. Since 1986, the ideas of Ashtekar blossomed and ramificated in various directions as shown by the extensive bibliography on this subject [33] The classical part is treated in detail by Romano. [34]. A short and pedagogical introduction tailored to our needs can be found in Giulini. [35]. Bianchi I and II were treated from a different point of view by Gonzalez and Tate. [36]. Relevant treatments of the classical part can also be found in the article of Manojlovic and Mikovic. [37] or in more details and background in the book of Ashtekar [38].

Our departure point is that Ashtekhar variables allow the simplification of the hamiltonian structure of Bianchi models to the point were they can be attacked by the methods of classical perturbation theory. These are almost as old as Newtonian dynamics itself. One of its uses was in celestial mechanics, where direct integration is not possible if more than two bodies are involved. The classic perturbation method [39], based on Hamilton-Jacobi technique shows divergences, namely,the infamous small divisors problems [41]. It was Poincaré who first realized the complexity of the solution in phase space [40]. In this vein, a great breakthrought was the Kalmagorov-Arnold-Moser (KAM) Theorem [41].

By now, the onset of chaos in weakly perturbed Hamiltonian systems is well understood. In this case, the Hamiltonian splits into two parts: the integrable part H_0 and the (small) perturbation ΔH. Usually, one makes a canonical transformation to action-angle variables$\left(\vec{I},\vec{\theta}\right)$. Then $H = H_0\left(\vec{I}\right) + \Delta H\left(\vec{I},\vec{\theta}\right)$. Unperturbed motion is restrained to tori on phase space and is either periodic or quasiperiodic (integrability means that there are $2N$ constants of motion, there being

$2N$ degrees of freedom. In the Hamiltonian case, N constants of motion are sufficient to ensure integrability since the other N follow trivially from Hamilton equations. These N constants of motion are given here by $\vec{I}$, which represents the radius of these tori). Expanding $\Delta H\left(\vec{I},\vec{\theta}\right)$ in Fourier series

$$\Delta H\left(\vec{I},\vec{\theta}\right)=\sum_{\vec{n}}\Delta H\left(\vec{I}\right)\exp\left[\vec{n}\cdot\vec{\theta}\right]$$

resonances will appear when $\vec{N}\cdot\vec{\omega}(I_{i0})=0$ where $\vec{N}$ are specific values for the $\vec{n}$, $\vec{\omega}$ are the unperturbed frecuencies and I_{i0} is some specific values of the actions $\vec{I}$. Trying to get rid of theses primary resonances leads to secondary resonances.By example a canonical transformation near one resonance will lead to a Hamiltonian of the form

$$\bar{H}=\sum_i \frac{d\omega_i(I_{i0})}{dI_i}\Delta I_i^2+\Delta H_0\cos\psi$$

where $\omega_i=\frac{dH(I_{i0})}{dI_i}$, $\Delta I_i=I_i-I_{i0}$ and $\psi=\vec{N}\cdot\vec{\theta}+\phi$. This is formally the Hamiltonian of the nonlinear pendulum which its own new resonance term and its structure of elliptic and hyperbolic fix points. That is, at a smaller scale in phase space, a whole new resonant structure appears [42] [41]. KAM theorem ensures that under sufficiently small perturbations, almost all tori are preserved, but those near the resonances are destroyed. Another usual feature is the existence of fixed point; the saddle points are of particular interest leading to the appearance of stable and unstable manifolds and of the so-called homoclinic (heteroclinoc if more than one saddle point are involved) chaos [41].

One difficulty with the Mixmaster dynamics is that periodic or quasiperiodic trajectories (in minisuperspace) do not exist and there is no fixed point. The absence of periodic or cuasiperiodic solutions is due to the monotonous increase (decrease) of the overall scale factor $\Omega\equiv\ln(abc)$ where a, b and c are the scale factors for the three axes. Indeed, it has been claimed that this absence would preclude chaotic behaviour for the Mixmaster [43]. But the monotonic increase (decrease) can be easily separated from the behaviour of the other significant variables and so one can recover quasiperiodic motions [27]. Solutions corresponding to a finite number of bounces before the trayectories make the perfect hit and go directly down one of the three channels are forbiden by the dynamics [44].

All this paper can be seen as an exercise in classical mechanics in the somewhat unusual context of General Relativity. Our goal is to investigate Bianchi IX using Ashtekar's formulation. We will solve the integrable Bianchi I and Bianchi II models using Hamilton-Jacobi. We will describe the Kasner epochs in this formulation and latter apply simple canonical perturbation theory viewing Bianchi IX as a perturbation of Bainchi II. We will identify exactly where this approach breaks down, and analize the reasons for this "failure".

II. BIANCHI MODELS AND ASHTEKAR'S VARIABLES

A. Homogeneous Cosmologies

Among the cosmological model of General Relativity, some are particularly interesting for their mathematical simplicity and their physical interest: the so-called Bianchi cosmological models [11]. Their simplicity resides in the fact that the spatial slice of these universe is homogeneous. The number of freedoms then reduces drastically and Einstein Equations become ordinary differential equations. The metric is

$$ds^2 = -N^2 dt^2 + q_{IJ}(t)\chi^I \otimes \chi^J \qquad ; \qquad I, J = 1, 2, 3 \tag{1}$$

Often this metric is written using the Misner's parametrization $q_{IJ} = \exp\left(-2\Omega(t)\right)\left(\exp 2\beta(t)\right)_{IJ}$ where $Tr(\beta) = 0$ .This metric is (left) invariant under (spatial) transformations generated by a certain group of symmetries that characterize each specific Bianchi model. As usual, the Killing vectors ξ_i are the infinitesimal generators of the isometries on these spaces . The left invariant vector fields L_I ($\chi^I L_J = \delta^I_J$) verify then

$$\pounds_{\xi_J} L_I = [\xi_I, L_J] = 0 \tag{2}$$

This also means that the Killing vectors are right invariant vector fields [47]. The left invariant vectors are related to the structure constants of the (Lie) group via

$$[L_I, L_J] = C^K{}_{IJ} L_K \tag{3}$$

The $C^K{}_{IJ}$ are the structure constants of the Lie group that leaves the metric q_{IJ} invariant. The spacetime spatial metric, that is, the metric written in a coordinate basis, is given by $q_{\mu\nu} = q_{IJ}\chi^I_\mu \chi^J_\nu$. General Relativity admits a Hamiltonian formulation. Since we are interested in homogeneous cosmologies, we would like to write a simpler Hamiltonian, one which is homogeneous from the start. The Hamilton equations obtained from this simpler Hamiltonian will be the correct ones (that is, the same as the one obtains using the full Hamiltonian of General Relativity and demanding homogeneity afterward [46] [11]), only when the structure constants verify

$$C^K{}_{IJ} = \epsilon_{IJL} S^{LK} \tag{4}$$

were ϵ_{IJK} is the antisymmetric tensor and S^{IJ} is a symmetric tensor. These models are called class A [11]. The simpler exemple is Bianchi I characterize by $C^K{}_{IJ} = 0$. The metric written in the Kasner form is:

$$ds^2 = t^{2p_1}\left(dx^1\right)^2 + t^{2p_2}\left(dx^2\right)^2 + t^{2p_3}\left(dx^3\right)^2 \tag{5}$$

with $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^3 = 1$. It is useful to use the following parametrization for the p's

$$p_1 = \frac{-u}{u^2 + u + 1} \tag{6}$$

$$p_2 = \frac{1 + u}{u^2 + u + 1} \tag{7}$$

$$p_3 = \frac{u\left(u + 1\right)}{u^2 + u + 1} \tag{8}$$

where we assume the following ordering $p_1 \leq p_2 \leq p_3$ and $1 \leq u \leq \infty$.

Another case of interest is Bianchi IX, $C^K{}_{IJ} = \epsilon^K{}_{IJ}$. Most of the evolution of Bianchi IX can be seen as a succession of Kasner epoch, that is the metric can be approximated to great accuracy by the Kasner metric. This stem from the fact that the potential consist of exponentially rising wall and are thus almost zero otherwise. Approximating these walls as vertical, the change from one Kasner solution to another one is given by the famous BKL u-map

$$u_{n+1} = \begin{cases} u_n - 1 & \text{if} \quad u_n \geq 2 \\ \left(u_n - 1\right)^{-1} & \text{if} \quad u_n \leq 2 \end{cases} \tag{9}$$

The $u_n > 2$ case are called epoch and $u_n < 2$ era. Chaotic behavior, if it happen, would be confined in the era change. This could be resumed by the following map

$$u_{N+1} = u_N - [u_N] \tag{10}$$

the Gauss map, relating one era with the next. Barrow proved this map to be chaotic [14].

B. Ashtekar formalism

We will make a brief sketch of Ashtekar formalism. We will work mainly using two basis for spacetime, an orthonormal basis $\{e_a\}$ and a coordinate basis $\{\partial_\alpha\}$. Greek indices refer to coordinate (sometimes called spacetime in this context) bases and latin indices to frames bases (sometimes called internal indices) . When taken from the beginning of the alphabet (α, $\beta, ..., a, b, ...$) their range is $\{0, 1, 2, 3\}$ whereas for the middle of the alphabet ($\mu, \nu, ..., i, j, ...$) their range is $\{1, 2, 3\}$.

The orthonormal basis and the metric are related through

$$\eta_{ab} e^a \otimes e^b = g_{\alpha\beta} dx^\alpha \otimes dx^\beta$$

$$\eta_{ab} e^a_\alpha e^b_\beta = g_{\alpha\beta}$$

Thus taking determinant ont both sides gives

$$e^2 \equiv \left(\det\left(e^a_\alpha\right)\right)^2 = \det\left(g_{\alpha\beta}\right) \equiv g$$

viewing e^a_α as a square matrix of order n .We will fix the dimensionality of spacetime $n = 4$. Let's fix an orthonormal base $\left\{{}^{(4)}e_\perp, {}^{(4)}e_i\right\}$ so that ${}^{(4)}e_\perp$ is normal to the hypersurface Σ. The dual co-tetrad is $\left\{{}^{(4)}e^\perp, {}^{(4)}e^i\right\}$. We have

$$\frac{\partial}{\partial t} = N e_\perp + N^i e_i$$

where N is the lapse function and $N^i e_i$ is the shift vector field. The relation between the spatial coordinate and frame basis is given by

$$\partial_\mu = e^i_\mu e_i \qquad ; \qquad e_i = e^\mu_i \partial_\mu$$

The point of departure is the action

$$S = \int (e) d^4x \, {}^+F^{ab}_{\alpha\beta}[A] e^\alpha_a e^\beta_b \tag{11}$$

where ${}^+F^{ab}_{\alpha\beta}[{}^{+(4)}A] = 2\partial_{[\alpha}\left({}^{+(4)}A_{\beta]}{}^{ab}\right) + 2\,{}^{+(4)}A_{[\alpha}{}^{a}{}_{|c|}\,{}^{+(4)}A_{\beta]}{}^{cb}$ are the self-dual two-form. ${}^{+(4)}A_\alpha{}^{ab}$ is a self-dual connection related to the ordinary (meaning neither self-dual or anti-self dual) connection by (see appendix 1)

$$\begin{aligned} {}^{+(4)}A_\alpha{}^{ab} &\equiv \frac{1}{2}\left({}^{(4)}A_\alpha{}^{ab} - \frac{1}{2}\epsilon^{ab}{}_{cd}\,{}^{(4)}A_\alpha{}^{cd}\right) && (12) \\ {}^{+(4)}A_\alpha{}^{ab} &= i^*\left({}^{+(4)}A_\alpha{}^{ab}\right) \\ &\equiv \frac{i}{2}\epsilon^{ab}{}_{cd}\,{}^{+(4)}A_\alpha{}^{cd} && (13) \end{aligned}$$

To recover the true degrees of freedom, as in conventional General Relativity, one makes an ADM decomposition ([7], see also [45]for an easier introduction; in this context, see [38]). The result is

$$\begin{aligned} S = \int \Big\{ &-i\epsilon^i{}_{jk}\partial_t\left(A_\mu{}^{jk}\right)\tilde{E}^\mu_i - D_\mu\left(i\epsilon^i{}_{jk}\tilde{E}^\mu_i\right)A_0{}^{jk} \\ &+ \underline{N}F^{ij}{}_{\mu\nu}\tilde{E}^\mu_i\tilde{E}^\nu_j + iN^\mu\epsilon^i{}_{jk}F^{jk}{}_{\mu\nu}\tilde{E}^\nu_i\Big\} d^3x dt \end{aligned} \tag{14}$$

where a surface term was dropped and D_μ is the covariant derivative constructed with $A_\mu{}^{ij}$ which lives exclusively in the spatial hypersurface Σ

$$D_\mu V_i = \partial_\mu V_i + A_{\mu i}{}^{j} V_j$$

ϵ^{ijk} is the totally antysimmetric tensor density and the following notation was introduced

$$\tilde{E}^{\mu}_{i} \equiv \sqrt{q} E^{\mu}_{i} = \frac{1}{\det\left(e^{\mu}_{i}\right)} \tag{15}$$

$$\underline{N} \equiv \frac{1}{\sqrt{q}} N = \det\left(e^{\mu}_{i}\right) N \tag{16}$$

$E^{\mu}_{i} \equiv \perp^{\mu}_{\nu} e^{\nu}_{i}$ is the projection of the tetrad in the hypersurface Σ. The action is thus written explicitly in the form $\int \left[p\dot{q} - H\right]$ whereas the Hamiltonian is a sum of constraints, since the conjugate momentum to the lapse, shift and the time-component of the 4-connection are absent. Our variables are $A_{\mu}{}^{jk}$ and $\Pi^{\mu}{}_{jk}$, the self-dual part of $-i\epsilon^{i}{}_{jk}\tilde{E}^{\mu}_{i}$. The constraint's equations then read

$$D_{\mu}\left(\epsilon^{i}{}_{jk}\tilde{E}^{\mu}_{i}\right) = 0 \tag{17}$$

$$F^{i}{}_{\mu\nu}\tilde{E}^{\nu}_{i} = 0 \tag{18}$$

$$\epsilon^{ij}{}_{k}F^{k}{}_{\mu\nu}\tilde{E}^{\mu}_{i}\tilde{E}^{\nu}_{j} = 0 \tag{19}$$

To write the constraints as above, we used the fact that the self-dual Lorentz algebra is isomorphic to the Lie algebra of complexified $SO\left(3\right)$. The isomorphism can be carried out by taking an internal vector n^{a} with $n^{a}n_{a} = -1$. Then every self-dual internal 2-form f_{ab} can be characterized completely by its "electric" part $2f_{ab}n^{a}$ which is an internal vector orthogonal to n^{a}. Hence

$$A^{a}_{\alpha} = 2iA^{\perp b}_{\alpha} \tag{20}$$

$$E^{\alpha}_{a} = 2\Pi^{\alpha}_{\perp b} \tag{21}$$

$$F^{a}{}_{\mu\nu} = 2iF^{\perp a}{}_{\mu\nu} \tag{22}$$

and we thus have

$$F^{i}{}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}{}^{i} + \epsilon^{i}{}_{jk}A_{[\mu}{}^{j}A_{\nu]}{}^{k} \tag{23}$$

To connect this formalism with Bianchi cosmologies let us introduce an orthonormal triad E_{i} and its co-triad basis E^{i}.

$$dl^{2} = \delta_{ij}E^{i}E^{j} \tag{24}$$

Our main interest in the introduction of this basis is to use them to define the Ashtekar's variables $\left(\tilde{E}^{\mu}_{i}, A^{i}_{\mu}\right)$

$$\tilde{E}_i^\mu \equiv \sqrt{\det\left(q_{\mu\nu}\right)} E_i^\mu \tag{25}$$

$$A_\mu^i \equiv \Gamma_\mu^i - iK_\mu^i \tag{26}$$

The tilde denotes a tensor density. The Γ_μ^i are related to the Ricci rotation coefficients and the K_μ^i to the extrinsic curvature. Note the presence of i.By construction, Ashtekar's variables (at least in this formulation [49]) are complex. We thus deal with a complex extension of General Relativity. In the end we will need to apply some "reality conditions " to ensure that the (spatial) metric constructed from the triad is real. Moreover one should also ask that the time evolution does not make the metric complex, so that the time derivative of the metric should also be real. Usually these variables are fields, that is they depend on $\vec{x}$ as well as on time t . In the homogeneous case, they will depend only on t . We can expand the triads on the invariant basis as follow

$$\tilde{E}_i^\mu = E_i^I L_I^\mu |\chi| \tag{27}$$

$$A_\mu^i = A_I^i \chi_\mu^I \tag{28}$$

were $|\chi| \equiv \det(\chi_\mu^I)$. Note that in this notation $E_i^I = E^{Ij}\delta_{ij}$ and $E_i^I = E_{Ji}q^{IJ}$ that is capital latin indices are raised and lowered using the invariant metric q_{IJ} and the lower case latin indices using the orthonormal metric δ_{ij} .We have the following relation

$$E_{Ii}E_J^i = q_{IJ}\det(E_i^I) \tag{29}$$

where $\det(E_i^I) = \det(q_{IJ})$.

We will be particularly interested in the diagonal case: $E_i^I \sim \delta_i^I$ and similarly for A_I^i. We will denote the diagonal variables as E_I and A_I respectively. The Hamiltonian constraint $H = 0$ reads

$$\begin{aligned} 0 &= \epsilon_i{}^{jk}\left(-A_I^i C^I{}_{JK} + \epsilon^i{}_{lm} A_J^l A_K^m\right) E_j^J E_k^K \\ &= \sum_{J,K} \epsilon_i{}^{jk}\left(-A_I C^I{}_{JK} + \epsilon^i{}_{jk} A_J A_K\right) E_J E_K \end{aligned} \tag{30}$$

We have the following relations

$$K_I \equiv K_I^i = \frac{1}{2\alpha}\frac{\partial}{\partial t}\left(\ln\left(q_I\right)\right)\frac{1}{E_I}\delta^{Ii} \tag{31}$$

$$\begin{aligned} \Gamma_I \equiv \Gamma_I^i = &-\epsilon^{NLP}\delta_{PI}\frac{1}{E_P}\left(\frac{1}{2}C_{\ NLP} - \frac{1}{4}C_{PNL}\right)\delta^{Ii} \\ &\left[= -\sum_{I=1}^{3}\epsilon^{NLI}\frac{1}{E_I}\left(\frac{1}{2}C_{\ NLI} - \frac{1}{4}C_{INL}\right)\right] \end{aligned} \tag{32}$$

$$A_I \equiv A^i_I = \frac{\omega_I}{E_I}\delta^{Ii} = \Gamma_I - iK_I \tag{33}$$

$$E_I = \sqrt{\frac{\det(q_I)}{q_I}} \tag{34}$$

The last relation can be easily inverted to give

$$q_D = \left|\frac{E_1E_2E_3}{E_D^2}\right| \tag{35}$$

III. BIANCHI I

As an introductory exercice and to explicitly relate this formalism to the usual Kasner solution, we will solve first the Bianchi I case. It is the simplest one, since all $C^K{}_{IJ} = 0$. Thus, the Hamiltonian is given by

$$H = (\ A_1A_2E_1E_2 + A_1A_3E_1E_3 + A_2A_3E_2E_3) \tag{36}$$

To find the equations of motion, in this case we could integrate quite easily the Hamilton equations. Instead, as a preparation fot the more complex case we will find a convenient canonical transformation that makes the integration trivial. Consider the following generating function that implement a change from the old (phase space) coordinates $\left(\vec{A},\vec{E}\right)$ to the new ones $\left(\vec{\beta},\vec{\omega}\right)$:

$$S(\vec{A},\vec{\omega}) = \omega_1 \ln A_1 + \omega_2 \ln A_2 + \omega_3 \ln A_3 - Et \tag{37}$$

Thus

$$E_i = \frac{\partial S}{\partial A^i} = \frac{\omega_i}{A^i} \tag{38}$$

$$\beta^i = \frac{\partial S}{\partial \omega_i} = \ln A^i - \frac{\partial E}{\partial \omega_i}t \tag{39}$$

and

$$H = (\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3) \tag{40}$$

implying that ω_i are constants of motion and the β^i are linear in the time parameter. Inverting we find

$$E_1 = \omega_1 \exp(-\beta_1)\exp[-(\omega_2+\omega_3)t] \tag{41}$$

$$E_2 = \omega_2 \exp(-\beta_2)\exp[-(\omega_1+\omega_3)t] \tag{42}$$

$$E_3 = \omega_3 \exp(-\beta_3)\exp[-(\omega_1+\omega_2)t] \tag{43}$$

Then

$$q_{11} = \frac{\omega_2\omega_3}{\omega_1} \exp\left(\beta_1 - \beta_2 - \beta_3\right) \exp\left(-2\omega_1\right) t \tag{44}$$

$$q_{22} = \frac{\omega_1\omega_3}{\omega_2} \exp\left(\beta_2 - \beta_1 - \beta_3\right) \exp\left(-2\omega_2\right) t \tag{45}$$

$$q_{33} = \frac{\omega_1\omega_2}{\omega_3} \exp\left(\beta_3 - \beta_1 - \beta_2\right) \exp\left(-2\omega_3\right) t \tag{46}$$

Up to now, our solution is complex. We have to apply the reality conditions. This means that both the metric and its time derivative must be real. This is easily seen to be obtained if

$$\omega_i = i\Omega_i \tag{47}$$

where Ω is real and

$$t = \mathrm{Re}[t] + i\,\mathrm{Im}[t] \tag{48}$$

$$= t_0 + i\ \mathrm{Im}[t] \tag{49}$$

t_0 a constant. We then obtain

$$q_{11} = \frac{\Omega_2\Omega_3}{\Omega_1} \exp\left[2\Omega_1\tilde{t}\right] \tag{50}$$

$$q_{22} = \frac{\Omega_1\Omega_3}{\Omega_2} \exp\left[2\Omega_2\tilde{t}\right] \tag{51}$$

$$q_{33} = \frac{\Omega_1\Omega_2}{\Omega_3} \exp\left[2\Omega_3\tilde{t}\right] \tag{52}$$

where we write $\tilde{t} \equiv \mathrm{Im}[t]$ and the prefactor can be absorbed upon a rescaling of the axes. Also, General Relativity imposes

$$H = 0 \tag{53}$$

then to recover General Relativity, the Ω 's should obey the following constraint:

$$\Omega_1\Omega_2 + \Omega_1\Omega_3 + \Omega_2\Omega_3 = 0 \tag{54}$$

For convenience, to connect with the usual analysis in term of bounces and eras later on [10], let us parametrize the Ω_i as follows

$$\Omega_1 = p_0 + p_+ + \sqrt{3}p_- \tag{55}$$

$$\Omega_2 = p_0 + p_+ - \sqrt{3}p_- \tag{56}$$

$$\Omega_3 = p_0 - 2p_+ \tag{57}$$

Using the Hamiltonian constraint, we find

$$p_+^2 + p_-^2 = p_0^2 \tag{58}$$

Let's introduce the angle θ :

$$p_+ = p_0 \cos\theta \tag{59}$$

$$p_- = p_0 \sin\theta \tag{60}$$

Let us define

$$\begin{aligned} \sigma_1 &\equiv \frac{\Omega_1}{\Omega_1+\Omega_2+\Omega_3} \\ &= \frac{2}{3}\left(\frac{1}{2} - \cos\left(\theta + \frac{2\pi}{3}\right)\right) \end{aligned} \tag{61}$$

$$\begin{aligned} \sigma_2 &\equiv \frac{\Omega_2}{\Omega_1+\Omega_2+\Omega_3} \\ &= \frac{2}{3}\left(\frac{1}{2} - \cos\left(\theta - \frac{2\pi}{3}\right)\right) \end{aligned} \tag{62}$$

$$\begin{aligned} \sigma_3 &\equiv \frac{\Omega_3}{\Omega_1+\Omega_2+\Omega_3} \\ &= \frac{2}{3}\left(\frac{1}{2} - \cos\theta\right) \end{aligned} \tag{63}$$

Note that $\sigma_1 + \sigma_2 + \sigma_3 = 1$ and $\sigma_1^2 + \sigma_2^2 + \sigma_3^3 = 1$. Redefining time

$$\tilde{t} \equiv \frac{1}{\Omega_1+\Omega_2+\Omega_3} \ln T \tag{64}$$

We obtain

$$q_{11} = \frac{\Omega_2\Omega_3}{\Omega_1} T^{2\sigma_1} \tag{65}$$

$$q_{22} = \frac{\Omega_1\Omega_3}{\Omega_2} T^{2\sigma_2} \tag{66}$$

$$q_{33} = \frac{\Omega_1\Omega_2}{\Omega_3} T^{2\sigma_3} \tag{67}$$

Which, upon rescaling of the coordinates $x^1 \to \grave{x}^1 = \Omega_1^{-1}\Omega_2\Omega_3 \ x^{\ 1}$, etc gives the well known Kasner's line element

$$ds^2 = T^{2\sigma_1}\left(dx^1\right)^2 + T^{2\sigma_2}\left(dx^2\right)^2 + T^{2\sigma_3}\left(dx^3\right)^2 \tag{68}$$

It is useful to divide the domain of the angle θ en six sectors

$$(j-1)\frac{\pi}{3} \le \theta \le j\frac{\pi}{3} \qquad ; \qquad j \in \{1,2,3,4,5,6\} \tag{69}$$

Another parametrisation of the σ_i is

$$\tilde{\sigma}_1 = \frac{-u}{u^2+u+1} \quad ; \quad \tilde{\sigma}_2 = \frac{1+u}{u^2+u+1} \quad ; \quad \tilde{\sigma}_3 = \frac{u(u+1)}{u^2+u+1} \tag{70}$$

where $1 \le u \le \infty$ and we write $\tilde{\sigma}$ to indicate the ordering $\tilde{\sigma}_1 \le \tilde{\sigma}_2 \le \tilde{\sigma}_3$. It is straigtforward to write the $u = u(\theta)$. The result is

$$j = 1 \quad , \quad u = \frac{1-\cos(\theta+\pi/3)}{\cos\theta - 1/2} \tag{71}$$

$$j = 2 \quad , \quad u = \frac{1+\cos\theta}{\cos(\theta-2\pi/3) - 1/2} \tag{72}$$

where the j even (odd) formulae are generated each from the preceeding one using $u(\theta) \to \bar{u}(\theta) = u(\theta - 2\pi/3)$.

IV. BIANCHI II

This model is characterized by $C^I{}_{JK} = \delta_3^I \epsilon_{3JK}$. The Hamiltonian is thus given by

$$H = (\ A_1A_2E_1E_2 + A_1A_3E_1E_3 + A_2A_3E_2E_3) - \frac{1}{G}A_3E_1E_2 \tag{73}$$

We introduce explicitly the Newton constant G. When this is done we have [48]

$$A^i_\mu \equiv \frac{1}{G}\left(\Gamma^i_\mu - iK^i_\mu\right)$$
$$F^i{}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}{}^i + G\ \epsilon^i{}_{jk}A_{[\mu}{}^jA_{\nu]}{}^k$$

As in the Bianchi I case, we begin with a canonical transformation generated by

$$F_2 = \ln\left[(GA_1)^{(G\omega_1)}(GA_2)^{(G\omega_2)}(GA_3)^{(G\omega_3)}\right] \tag{74}$$

thus

$$H = (\ \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3) - \omega_1\omega_2 \exp\frac{(\beta_3 - \beta_1 - \beta_2)}{G} \tag{75}$$

Since every one dimensional problem is integrable and since the β appears in a very special manner in the Hamiltonian, we propose a second canonical transformation, this time generated by

$$G_2 = \beta_1 I_1 + \beta_2 I_2 + (\beta_3 - \beta_1 - \beta_2)\, I_3 \tag{76}$$

We will then go from $\left(\vec{\beta}, \vec{\omega}\right) \to \left(\vec{\gamma}, \vec{I}\right)$. That will effectively convert the hamiltonian to an equivalent unidimensional one

$$H=\left(I_1I_2-I_3^2\right)-\left[I_1I_2-\left(I_1+I_2\right)I_3+I_3^2\right]\exp\frac{\gamma_3}{G} \tag{77}$$

Again, with an eye on future complications, we obtain the last canonical transformation to reduce this problem to a trivial one by applying the well know Hamilton-Jacobi technique. The generating function that will finally resolve our system, S [50]

$$S=W(\gamma_3,P_1,P_2,E)+\gamma_1P_1+\gamma_2P_2-Et \tag{78}$$

is a solution of the Hamilton-Jacobi equation

$$H\left(P_1,P_{2,}\frac{\partial W}{\partial\gamma_3},\gamma_3\right)+\frac{\partial S}{\partial t}=0 \tag{79}$$

that is

$$E=\left(P_1P_2-\left(\frac{\partial W}{\partial\gamma_3}\right)^2\right) \tag{80}$$

$$-\left[P_1P_2-\left(P_1+P_2\right)\frac{\partial W}{\partial\gamma_3}+\left(\frac{\partial W}{\partial\gamma_3}\right)^2\right]\exp\frac{\gamma_3}{G} \tag{81}$$

the other factors $P_1$and P_2 are constants. The solution is

$$W\ =\frac{G}{2}\left(P_1+P_2\right)\int\frac{dx}{(x+1)}-\frac{G}{2}\epsilon\int\frac{\sqrt{cx^2+bx+a}}{x\left(x+1\right)}dx \tag{82}$$

where ϵ is ±1

$$c\equiv\left(P_1-P_2\right)^2 \tag{83}$$

$$b\equiv-4E \tag{84}$$

$$a\equiv4(P_1P_2-E) \tag{85}$$

and we define

$$x\equiv\exp\frac{\gamma_3}{G} \tag{86}$$

Now

$$\theta_3=\frac{\partial W}{\partial E}-t \tag{87}$$

$$=\frac{-G}{\sqrt{a}}\epsilon\ \ln\left[\frac{2\sqrt{a}\sqrt{R(x)}+bx+2a}{\sqrt{4ac-b^2}x}\right]-t \tag{88}$$

where we write $R(x) = cx^2 + bx + a$ to abreviate the notation. Thus

$$K \exp\left(-\sqrt{a}\epsilon\frac{(t+\theta_3)}{G}\right) = \frac{2\sqrt{a}\sqrt{R(x)} + 2a + bx}{x} \tag{89}$$

with

$$K \equiv \sqrt{4ac - b^2} \tag{90}$$

Note that K posses units of E (same as units of I_i^2) . Writing

$$u \equiv \exp\left(-\sqrt{a}\epsilon\frac{(t+\theta_3)}{G}\right) \tag{91}$$

and solving for x, we obtain

$$x = \frac{4aKu}{(Ku-b)^2 - 4ac} \tag{92}$$

We can now compute I_3

$$I_3 = \frac{\partial W}{\partial \gamma_3} \tag{93}$$

$$= \frac{-1}{2}\epsilon\sqrt{a}\frac{(Ku - 4\epsilon\gamma)(Ku + 4\epsilon\lambda)}{(Ku + 4\gamma)(Ku + 4\lambda)} \tag{94}$$

where we use the definitions

$$\gamma = (2P_1P_2 - E) + (P_1 + P_2)\sqrt{P_1P_2 - E} \tag{95}$$

$$\lambda = (2P_1P_2 - E) - (P_1 + P_2)\sqrt{P_1P_2 - E} \tag{96}$$

The next step is to find θ_1 and θ_2.

$$\theta_1 = \frac{\partial S}{\partial P_1} \tag{97}$$

$$= \gamma_1 + G\ln\frac{(Ku+4\gamma)}{(Ku+4\beta)} + \frac{GP_2}{\sqrt{a}}\ln u \tag{98}$$

similarly

$$\theta_2 = \gamma_2 + G\ln\frac{(Ku+4\gamma)}{(Ku-4\alpha)} + \frac{GP_1}{\sqrt{a}}\ln u \tag{99}$$

were we make the choice $\epsilon = -1$ (the $\epsilon = -1$ case leads to an equivalent solution) and we used the following definitions

$$\alpha = (P_1 - P_2)\sqrt{P_1P_2 - E} - E \tag{100}$$

$$\beta = (P_1 - P_2)\sqrt{P_1P_2 - E} + E \tag{101}$$

We are now able to compute the original Ashtekar's variables. Let us then compute the E_i.

$$E_1 = \frac{1}{4G}\left(P_1 + \frac{1}{2}\sqrt{a}\right)\exp\left(-\frac{\theta_1}{G}\right)\exp\left(-\frac{P_2(t+\theta_3)}{G}\right) \tag{102}$$

$$E_2 = \frac{1}{4G}\left(P_2 + \frac{1}{2}\sqrt{a}\right)\exp\left(-\frac{\theta_2}{G}\right)\exp\left(-\frac{P_1(t+\theta_3)}{G}\right) \tag{103}$$

$$\begin{aligned} E_3 = &-\frac{1}{8G\sqrt{a}}\frac{(K^2u^2 - 16\gamma^2)}{Ku}\times \\ &\times\exp\left(-\left(\frac{\theta_1+\theta_2}{G}\right)\right)\exp\left(-\left(\frac{P_1+P_2}{G}\right)(t+\theta_3)\right) \end{aligned} \tag{104}$$

Writing down the metric is straightforward, since $q_{11} = \frac{E_1E_2E_3}{E_1^2}$, and similarly the other components.

Up to now, all our work was made with the asumption that the variables were complex. To connect with ordinary general relativity, we have to ask that q_{ii} and $\dot{q}_{ii}$ should be real. But the A^i should be complex since $\Gamma^i - iK^i$. Let us then choose

$$P_1 \equiv ip_1 \qquad , \qquad p_1 \in \mathrm{Re}^+ \tag{105}$$

$$P_2 \equiv ip_2 \qquad , \qquad p_2 \in \mathrm{Re}^+ \tag{106}$$

$$E, K \in \mathrm{Re}^+ \tag{107}$$

The hamiltonian constraint forces us to choose

$$E = 0 \tag{108}$$

thus

$$\sqrt{a} = 2\sqrt{P_1P_2} \tag{109}$$

$$\alpha = \sqrt{P_1P_2}\,(P_1 - P_2) \tag{110}$$

$$\gamma = \sqrt{P_1P_2}(\sqrt{P_1} + \sqrt{P_2})^2 \tag{111}$$

To understand qualitatively the evolution let us focus on the essential features of the dynamics. Without the multiplicative constants, we have:

$$E_1 = \exp\left(-\frac{p_2t}{G}\right) \tag{112}$$

$$E_2 = \exp\left(-\frac{p_1t}{G}\right) \tag{113}$$

$$E_3 = \frac{(K^2u^2 - 16\gamma^2)}{Ku}\exp\left(-\left(\frac{p_1+p_2}{G}\right)t\right) \tag{114}$$

where $u \equiv \exp\left(-\sqrt{a}G^{-1}t\right)$. In the limit $t \to -\infty$, we have

$$E_1 = \exp\left(-\frac{p_2 t}{G}\right) \tag{115}$$

$$E_2 = \exp\left(-\frac{p_1 t}{G}\right) \tag{116}$$

$$E_3 = \exp\left(-\frac{\sqrt{p_1 p_2}}{G}t\right)\exp\left(-\left(\frac{p_1+p_2}{G}\right)t\right) \tag{117}$$

In the limit $t \to \infty$, we have

$$E_1 = \exp\left(-\frac{p_2 t}{G}\right) \tag{118}$$

$$E_2 = \exp\left(-\frac{p_1 t}{G}\right) \tag{119}$$

$$E_3 = \exp\left(\frac{\sqrt{p_1 p_2}}{G}t\right)\exp\left(-\left(\frac{p_1+p_2}{G}\right)t\right) \tag{120}$$

These asymptotic forms for E_i have the typical aspect of a Bianchi I solution

$$E_1 = \exp\left[-(\Omega_2 + \Omega_3)t\right] \tag{121}$$

$$E_2 = \exp\left[-(\Omega_1 + \Omega_3)t\right] \tag{122}$$

$$E_3 = \exp\left[-(\Omega_1 + \Omega_2)t\right] \tag{123}$$

and

$$E_1 = \exp\left[-(\Omega_2^N + \Omega_3^N)t\right] \tag{124}$$

$$E_2 = \exp\left[-(\Omega_1^N + \Omega_3^N)t\right] \tag{125}$$

$$E_3 = \exp\left[-(\Omega_1^N + \Omega_2^N)t\right] \tag{126}$$

thus we can approximate the evolution as a transition from one Bianchi I-like state to another (the famous "bounce" in the usual minisuperspace version of homogeneous cosmology, where the evolution is seen as the motion of a particle in a potential [11]). The transition from one Bianchi I history to another is better understood if one redefines new $\tilde{\Omega}_i$ and $\tilde{\Omega}_i^N$ such that

$$\tilde{\Omega}_1 \leq \tilde{\Omega}_2 \leq \tilde{\Omega}_3 \tag{127}$$

$$\tilde{\Omega}_1^N \leq \tilde{\Omega}_2^N \leq \tilde{\Omega}_3^N \tag{128}$$

Then

$$\Omega_1^N = \Omega_1 + 2\Omega_3 \tag{129}$$

$$\Omega_2^N = \Omega_2 + 2\Omega_3 \tag{130}$$

$$\Omega_3^N = -\Omega_3 \tag{131}$$

Introducing a new angle θ^N

$$p_+^N = p_0^N \cos\theta^N \tag{132}$$

$$p_-^N = p_0^N \sin\theta^N \tag{133}$$

we can easily find the transformation law for the parameter θ

$$\cos\theta^N = \frac{4 - 5\cos\theta}{5 - 4\cos\theta} \tag{134}$$

$$\sin\theta^N = \frac{3\sin\theta}{5 - 4\cos\theta} \tag{135}$$

This is of course essentially the BKL [10] analysis seen in a somewhat unfamiliar context.

V. BIANCHI IX

In this case, we will see that a straightforward application of classical perturbation theory fails to find a canonical transformation that makes the Hamiltonian trivial and thus the integration possible. This failure is of course perfectly understandable in the light of recent developments concerning Bianchi IX [25] [26] [27].The Hamiltonian is

$$\begin{aligned} H = (\ & A_1A_2E_1E_2 + A_1A_3E_1E_3 + A_2A_3E_2E_3) - \\ & \frac{1}{G}(A_3E_1E_2 + A_1E_2E_3 + A_2E_3E_1) \end{aligned} \tag{136}$$

By inspection, one recognizes part of this Hamiltonian as the Bianchi II that was shown previously to be integrable. Using the same canonical transformation as before, namely equations (74) and (76), we find

$$\begin{aligned} H = & \left(I_1I_2 - I_3^2\right) - \left[I_1I_2 - (I_1 + I_2)\, I_3 + I_3^2\right] \exp\frac{\gamma_3}{G} \\ & -(I_1 - I_3)I_3 \exp\frac{-\left(2\gamma_1 + \gamma_3\right)}{G} - (I_2 - I_3)I_3 \exp\frac{-\left(2\gamma_2 + \gamma_3\right)}{G} \end{aligned} \tag{137}$$

With a further transformation

$$S_2 = W(\gamma_3, P_1, P_2, E) + \gamma_1P_1 + \gamma_2P_2 \tag{138}$$

using (100, 101, 95 and 96) one finds

$$I_1 - I_3 = \left(P_1 + \frac{1}{2}\sqrt{a}\right)\left(\frac{Ku + 4\beta}{Ku + 4\gamma}\right) \tag{139}$$

$$I_2 - I_3 = \left(P_2 + \frac{1}{2}\sqrt{a}\right)\left(\frac{Ku - 4\alpha}{Ku + 4\gamma}\right) \tag{140}$$

$$\begin{aligned} H = E & \\ & + \frac{1}{8\sqrt{a}}\frac{(Ku - 4\gamma)}{Ku}\left\{\left(P_1 + \frac{1}{2}\sqrt{a}\right)(Ku - 4\alpha)\exp\frac{-2\theta_1}{G}\exp\frac{-2P_2\theta_3}{G}\right. \\ & \left. + \left(P_2 + \frac{1}{2}\sqrt{a}\right)(Ku + 4\beta)\exp\frac{-2\theta_2}{G}\exp\frac{-2P_1\theta_3}{G}\right\} \end{aligned} \tag{141}$$

We know that General Relativity will require $H = 0$. Since we want to consider a weak perturbation, we can set $E = 0$ in the perturbed Hamiltonian.

$$\sqrt{a} = 2\sqrt{P_1 P_2} \tag{142}$$

$$\alpha = \sqrt{P_1 P_2}\,(P_1 - P_2) \tag{143}$$

$$\gamma = \sqrt{P_1 P_2}(\sqrt{P_1} + \sqrt{P_2})^2 \tag{144}$$

$$u = \exp\left(-2\sqrt{P_1 P_2}\frac{\theta_3}{G}\right) \tag{145}$$

$$K = 4\sqrt{P_1 P_2}\,(P_1 - P_2) \tag{146}$$

Thus

$$\begin{aligned} H = E + \frac{1}{4}\Bigg\{ & \left[\sqrt{P_1}\left(\sqrt{P_1} + \sqrt{P_2}\right)(P_1 - P_2)\,u - 2P_1\left(\sqrt{P_1} + \sqrt{P_2}\right)^2\right. \\ & \left. + \sqrt{P_1}\left(\sqrt{P_1} + \sqrt{P_2}\right)^3 u^{-1}\right]\exp\frac{-2\theta_1}{G}\exp\frac{-2P_2\theta_3}{G} \\ & + \left[\sqrt{P_2}\left(\sqrt{P_1} + \sqrt{P_2}\right)(P_1 - P_2)\,u - 2P_2\left(\sqrt{P_1} + \sqrt{P_2}\right)^2\right. \\ & \left. - \sqrt{P_2}\left(\sqrt{P_1} + \sqrt{P_2}\right)^3 u^{-1}\right]\exp\frac{-2\theta_2}{G}\exp\frac{-2P_1\theta_3}{G}\Bigg\} \end{aligned} \tag{147}$$

We then have the following Hamiltonian

$$H(\vec{P}, \vec{\theta}) = H_0(\vec{P}) + \triangle H(\vec{P}, \vec{\theta}) \tag{148}$$

We are looking for a generating function F:

$$F = \vec{\theta} \cdot \vec{J} + X(\vec{\theta}, \vec{J}) \tag{149}$$

so that

$$H(\vec{P},\vec{\theta}) \to H_0(\vec{J}) \tag{150}$$

up to first order. Thus F generates a canonical transformation from $\left(\vec{\theta},\vec{P}\right)$ to $\left(\vec{\phi},\vec{J}\right)$.We want to integrate the system up to first order in the perturbation. We know that

$$P_i = \frac{\partial F_2}{\partial \theta^i} = J_i + \frac{\partial X}{\partial \theta^i} \tag{151}$$

$$\phi^i = \frac{\partial F_2}{\partial J_i} = \theta^i + \frac{\partial X}{\partial J_i} \tag{152}$$

By Taylor

$$H(\vec{P},\vec{\theta}) = H(\vec{J},\vec{\phi}) + \frac{\partial H}{\partial \theta^i}\left(\theta^i - \phi^i\right) + \frac{\partial H}{\partial P_i}\left(P_i - J_i\right) + O\left(\epsilon^2\right) \tag{153}$$

$$= H_0(\vec{J}) + \triangle H(\vec{J},\vec{\phi}) + \frac{\partial H_0}{\partial I_i}\frac{\partial X}{\partial \theta^i} + O\left(\epsilon^2\right) \tag{154}$$

Our case is even simpler since

$$H_0 = E\,(\equiv P_3) \tag{155}$$

Thus

$$H(\vec{P},\vec{\theta}) = H_0(\vec{J}) + \triangle H(\vec{J},\vec{\phi}) + \frac{\partial X}{\partial \theta_3} + O\left(\epsilon^2\right) \tag{156}$$

If X is so that

$$H(\vec{P},\vec{\theta}) \to H_0(\vec{J}) + O\left(\epsilon^2\right) \tag{157}$$

then

$$\triangle\, H(\vec{J},\vec{\phi}) + \frac{\partial X}{\partial \theta_3} = O\left(\epsilon^2\right) \tag{158}$$

that is

$$\frac{\partial X}{\partial \theta_3} = -\triangle\, H(\vec{J},\vec{\phi}) + O\left(\epsilon^2\right) \tag{159}$$

which is the same as asking that

$$\frac{\partial X}{\partial \theta_3} = -\triangle\, H(\vec{J},\vec{\theta}) + O\left(\epsilon^2\right) \tag{160}$$

or

$$X = -\int \triangle H(\vec{J},\vec{\theta})d\theta_3 + O\left(\epsilon^2\right) \tag{161}$$

Thus

$$\begin{aligned} X = \frac{G}{8}\Bigg\{ & \Bigg[\frac{\sqrt{P_1}\left(P_1-P_2\right)}{\sqrt{P_2}}\exp\frac{-2\sqrt{P_2}\left(\sqrt{P_1}+\sqrt{P_2}\right)\theta_3}{G} \\ & \frac{-2P_1\left(\sqrt{P_1}+\sqrt{P_2}\right)^2}{P_2}\exp\frac{-2P_2\theta_3}{G} \\ & -\frac{\sqrt{P_1}\left(\sqrt{P_1}+\sqrt{P_2}\right)^3}{\sqrt{P_2}\left(\sqrt{P_1}-\sqrt{P_2}\right)}\exp\frac{-2\sqrt{P_2}\left(\sqrt{P_2}-\sqrt{P_1}\right)\theta_3}{G}\Bigg]\exp\frac{-2\theta_1}{G} \\ & +\Bigg[\frac{\sqrt{P_2}\left(P_1-P_2\right)}{\sqrt{P_1}}\exp\frac{-2\sqrt{P_1}\left(\sqrt{P_1}+\sqrt{P_2}\right)\theta_3}{G}-\frac{2P_2\left(\sqrt{P_1}+\sqrt{P_2}\right)^2}{P_1}\exp\frac{-2P_1\theta_3}{G} \\ & -\frac{\sqrt{P_2}\left(\sqrt{P_1}+\sqrt{P_2}\right)^3}{\sqrt{P_1}\left(\sqrt{P_1}-\sqrt{P_2}\right)}\exp\frac{-2\sqrt{P_1}\left(\sqrt{P_1}-\sqrt{P_2}\right)\theta_3}{G}\Bigg]\exp\frac{-2\theta_2}{G}\Bigg\} \end{aligned} \tag{162}$$

is the generating function that would permit the integration of the whole system albeit to first order. But the generating function X diverges and is thus useless near the following planes

$$P_1 = 0 \tag{163}$$

$$P_2 = 0 \tag{164}$$

$$P_1 = P_2 \tag{165}$$

The physical significance of these values is explained in the appendix 2.

Let us analyse the $P_2 = 0$ resonance. It is clear that the problem with X arises because the exponents lead to inverse powers of P_2 in the generating function; positive powers of P_2 in the pre exponential factors only improve integrability; thus, to analyze whether the system may be integrated or not, we may drop all powers of P_2 from the prefactors and write H as

$$\begin{aligned} H &= E + \frac{1}{4}P_1^2\exp\frac{-2\theta_1}{G}\exp\frac{-2P_2\theta_3}{G}\left\{-2+\exp\frac{-2\sqrt{P_2}\sqrt{P_1}\theta_3}{G}+\exp\frac{2\sqrt{P_2}\sqrt{P_1}\theta_3}{G}\right\} \\ &= E + \frac{1}{2}P_1^2\exp\frac{-2\theta_1}{G}\exp\frac{-2P_2\theta_3}{G}\left\{\cosh\frac{2\sqrt{P_2}\sqrt{P_1}\theta_3}{G}-1\right\} \end{aligned} \tag{166}$$

The typical (perturbative) route to chaos is seen by seeking second order resonances. Let's then integrate the more resonant term using the following generating function

$$S_2 = G\alpha\exp\left(\frac{\theta_1}{G}\right) + \varepsilon\theta_2 + K\theta_3 - \frac{1}{4}\frac{\alpha^2 G}{\varepsilon}\left(\exp\left[-2\frac{\varepsilon\theta_3}{G}\right]-1\right) \tag{167}$$

which allows us to make the change from $(\theta_1, \theta_2, \theta_3, P_1, P_2, E) \to (\psi_1, \psi_2, \psi_3, \alpha, \varepsilon, K)$. The new Hamiltonian reads

$$H = K + \frac{1}{2}\alpha_1^2 \exp\frac{-2\varepsilon\psi_3}{G} \cosh\left\{\frac{2\sqrt{\varepsilon}\psi_3}{G}\sqrt{\frac{\alpha}{G}\psi_1 + \frac{\alpha^2}{2\varepsilon}\left(\exp\frac{-2\varepsilon\psi_3}{G} - 1\right)}\right\} \tag{168}$$

where

$$K = E + \frac{1}{2}P_1^2 \exp\frac{-2\theta_1}{G} \exp\frac{-2P_2\theta_3}{G} \tag{169}$$

While we seem to have isolated the resonance at $\varepsilon = 0$, the new Hamiltonian has other secundary resonances. To bring them forth, let us attempt to integrate the new Hamiltonian to first orden using

$$X = -\int \triangle K d\psi_3 \tag{170}$$

$$= -\frac{G\alpha^2}{4\varepsilon}\int \exp(-z)\cosh\left\{z\sqrt{\hat{A} + \hat{B}\exp(-z)}\right\} dz \tag{171}$$

where

$$\hat{A} = \frac{\alpha}{\varepsilon G}\psi_1 - \frac{\alpha^2}{2\varepsilon^2} = \frac{\alpha}{\varepsilon}\left(\frac{\psi_1}{G} - \frac{\alpha}{2\varepsilon}\right) \tag{172}$$

$$\hat{B} = \frac{\alpha^2}{2\varepsilon^2} \tag{173}$$

Consider

$$\begin{aligned} I_+ &= \int \exp(-z)\exp\left(zA\sqrt{1 + B\exp(-z)}\right) dz \\ &= \int \exp\left[\left(-1 + A\sqrt{1 + B\exp(-z)}\right) z\right] \end{aligned} \tag{174}$$

$$\begin{aligned} I_- &= \int \exp(-z)\exp\left(-zA\sqrt{+B\exp(-z)}\right) dz \\ &= \int \exp\left[-\left(1 + A\sqrt{1 + B\exp(-z)}\right) z\right] \end{aligned} \tag{175}$$

where

$$A = \sqrt{\hat{A}} = \sqrt{\frac{\alpha}{\varepsilon}\left(\frac{\psi_1}{G} - \frac{\alpha}{2\varepsilon}\right)} \tag{176}$$

$$B = \frac{\hat{B}}{\hat{A}} = \frac{\alpha}{2\varepsilon\left(\frac{\psi_1}{G} - \frac{\alpha}{2\varepsilon}\right)} \tag{177}$$

Thus

$$X = -\frac{G\alpha^2}{8\varepsilon}(I_+ + I_-) \tag{178}$$

Let's analyse the divergence of $I_\pm$. Let's tackle the case of I_+ first.

$$I_+(A) = \sum_{k=0}^{\infty} I_k A^k \tag{179}$$

where

$$\begin{aligned} I_k &= \frac{1}{2\pi i} \oint \frac{1}{A^{k+1}} \int dz \exp\left[\left(-1 + A\sqrt{1 + B\exp(-z)}\right) z\right] \\ &= \frac{1}{k!} \int_0^\infty x^k \exp(-x) \left(1 + B\exp(-x)\right)^{k/2} dx \qquad (180) \\ &\simeq \left(1 + B\exp(-k)\right)^{k/2} \qquad (181) \end{aligned}$$

The series diverges when $A = 1$. We argue that $A = 1$ is a (simple) pole for this series, at least for some range of the initial parameters. To show this, we will construct an analytic continuation for the series (see appendix 3). Then

$$I_+ = \sum_{k=0}^{\infty} A^k + \sum_{k=0}^{\infty} F_k(B) \left(\frac{A}{2}\right)^k \tag{182}$$

where

$$F_k(B) \leq \frac{1}{4} B \frac{\left(\sqrt{1+B}\right)^k - 1}{\sqrt{1+B} - 1} \tag{183}$$

We argue that $A = 1$ is a simple pole since the series

$$\sum_{k=0}^{\infty} F_k(B) \left(\frac{A}{2}\right)^k \tag{184}$$

are analytic in $A = 1$. Using equation 176 we encounter "resonance" for the values

$$\frac{\alpha}{\varepsilon}\left(\frac{\psi_1}{G} - \frac{\alpha}{2\varepsilon}\right) = 1 \tag{185}$$

That is

$$\epsilon = \frac{1}{2} P_1 \pm \frac{1}{2}\sqrt{P_1^2 - 2\alpha} \tag{186}$$

where we used the generating function 167 to obtain the relation between α, ψ_1 and P_1

$$\frac{\alpha}{G}\psi_1 = P_1 \tag{187}$$

VI. CONCLUSION

Using Ashtekar formalism, we treat the Bianchi IX Hamiltonian using the tools of classical perturbation theory. Namely, we treat Bianchi IX as a perturbation of the integrable Bianchi II. The failure to integrate Bianchi IX perturbatively did not come as a surprise in view of the numerous analytical and numerical evidences of its chaotic character. Our main purpose was to show how the use of Ashtekar variable simplify the perturbative analysis since the Hamiltonian (often refered to as the scalar constraint in this context) is notably simpler then in the usual one. As a by product of our analysis we filled some gaps in the literature, such us recovering the BKL map in this context. The reality conditions, which are often difficult to handle are easily deal with here using the known Bianchi II solution and the relation between the two formalisms.

From the point of view of the larger framework of chaos and general relativity, the important point is to analyze the reasons for the "failure" of our naive approach to Bianchi IX. Our strategy was simply to apply to Bianchi IX, once rendered manageable by the translation into Ashtekhars variables, the most direct known ways to handle a Hamiltonian system. We treaded at first on known grounds, and not surprisingly found a quick success (providing a more complete solution to the Bianchi II case than previously reported, a tribute to the power of Hamilton - Jacobi methods). Thus encouraged, we faced the Bianchi IX problem.

The expectation in facing a complex Hamiltonian dynamical system, but that can be divided into an integrable part and a "perturbation", is that either no resonances will appear, and then the system will be integrable by virtue of KAM theory, or else there will be resonances. If these are isolated, however, the dynamics is not yet chaotic, but rather the resonances appear as boundaries of regions of integrability. Chaos arises when resonances appear in layers, an infinite set of "secondary" resonances accumulating towards the primary ones.

Because of the complex nature of Ashtekhar variables, it could not be expected that this analysis would translate in any straightforwrad way to our problem; rather, our aim has been to discern whether a similar structure to the familiar weak chaos appeared in our problem. Thus, instead of primary resonances, we found that naive perturbation theory (seeing Bianchi IX as a perturbation of Bianchi II) breaks down in channel runs, and also near the singular solutions at the end of the channels. The issue then became whether these were isolated singularities, or rather there were other breakdown points arbitrarily close to them.

In order to find an answer, we zeroed on the $P_2 = 0$ singular metric, by isolating the terms responsible for the breakdown of naive perturbation theory, and proceeded to the exact integration of this most

singular part of the Hamiltonian. If (to our surprise) no further obstructions to integrability had appeared, this would have been an indicator of a non chaotic nature of the whole Bianchi IX system; alas, our expectations held on, and we found that, arbitrarily close to the original singular solution, new singularities of perturbation theory appeared, as given by Eqs. (186) and (187).

To analyze these conditions, we must remember that, from the point of view of the dynamics generated by the Hamiltonian K, both α and ψ_1 are constants of motion. Thus what we found is, for any given value of the original momentum P_1, an hyperbola of trajectories where perturbation theory breaks down, approaching asymptotically the known singularity at $\varepsilon = 0$ when $\alpha \to 0$. The $P_2 = \varepsilon = 0$ singular point cannot be isolated, therefore, and we must expect complex behavior in a neigborhood of the corners of Misners triangular potential well.

We see that the failure of our naive approach in fact is leading us directly to the sources of complexity in the dynamics. This bolsters the conclusion that Ashtekar variables are a useful tool in studying chaotic behavior in General Relativity.

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VII. APPENDIX 1

We will resume here some basic notions of differential geometry. Our notation follows [45]. First, given two n-forms λ and σ at some point in M :

$$\lambda = \frac{1}{n!}\lambda_{a_1 \ldots a_n} e^{a_1} \wedge \ldots \wedge e^{a_n} \tag{188}$$

$$\sigma = \frac{1}{n!}\sigma_{a_1 \ldots a_n} e^{a_1} \wedge \ldots \wedge e^{a_n} \tag{189}$$

we defined their inner product, induced by g , via

$$< \lambda, \sigma > \equiv \frac{1}{n!}\lambda_{a_1 \ldots a_n}\ g^{a_1 b_1} \ldots g^{a_n b_n} \sigma_{b_1 \ldots b_n} = \lambda_{a_1 \ldots a_n} \sigma^{a_1 \ldots a_n} \tag{190}$$

The following proposition follows readily

$$\lambda \wedge^{\star} \sigma =< \lambda, \sigma > \epsilon \tag{191}$$

where ϵ is the volume form on M induced by the metric

$$\epsilon = \sqrt{g}dx^0 \wedge ... \wedge dx^{(n-1)} \tag{192}$$

$$= \frac{1}{n!}\epsilon_{a_1...a_n}\ e^{a_1} \wedge ... \wedge e^{a_n} \tag{193}$$

where $\epsilon_{a_1...a_n} \equiv n!\delta^0_{[a_1}...\delta^{(n-1)}_{a_n]}$.

Consider the following string of equalities

$$\begin{aligned} S &= \int F_{ab} \wedge^* \left(e^a \wedge e^b\right) \\ &= \frac{1}{2}\int R_{abcd}\left(e^c \wedge e^d\right) \wedge^* \left(e^a \wedge e^b\right) \\ &= \frac{1}{2}\int R_{abcd} < e^c \wedge e^d, e^a \wedge e^b > \epsilon \\ &= \frac{1}{2}\int R_{abcd}\left(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}\right)\epsilon \\ &= \int R\epsilon \end{aligned} \tag{194}$$

This show how to write the Einstein-Hilbert action in terms of the curvature 2-form F_{ab} and the co-tetrads e^a . The curvature F_{ab} is defined as

$$F_{ab} \equiv \frac{1}{2}R_{abcd}\ e^c \wedge e^d \tag{195}$$

and satisfy the second Cartan 's structure equation

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = F_{ab} \tag{196}$$

The $\omega^a{}_b$ are the connection 1-forms, defined as

$$\omega^a{}_b \equiv \Gamma^a_{bc}e^c \tag{197}$$

The component of the $\omega^a{}_b$ in coordinate base are called the Ricci rotation coefficients [47]

$$\Gamma^a_{bc}e^c_\alpha dx^\alpha = \omega_\alpha{}^a{}_b dx^\alpha \tag{198}$$

That is

$$\omega_\alpha{}^a{}_b = e^c_\alpha\Gamma^a_{bc} \tag{199}$$

It is straightforward to prove that $\omega_{ab} = -\omega_{ba}$ [47]. The following definition is useful

$$
\begin{aligned}
\nabla_\alpha V &= \nabla_\alpha (V_a e^a) \\
&= \partial_\alpha (V_a) e^a + V_a e^b_\alpha \nabla_b e^a \\
&= (\partial_\alpha (V_a) - V_c \omega_\alpha{}^c{}_a) e^a \\
&= (\partial_\alpha (V_a) + V_c \omega_{\alpha a}{}^c) e^a \equiv D_\alpha (V_a) e^a
\end{aligned} \tag{200}
$$

Finally the action can be rewritten as follows

$$
\begin{aligned}
S &= \int F_{ab} \wedge^* \left(e^a \wedge e^b\right) \\
&= \frac{1}{2} \int R_{abcd} \left(e^c \wedge e^d\right) \wedge^* \left(e^a \wedge e^b\right) \\
&= \frac{1}{2} \int R_{ab\alpha\beta} e^\alpha_c e^\beta_d \left(e^c \wedge e^d\right) \wedge^* \left(e^a \wedge e^b\right) \\
&= \frac{1}{2} \int R_{ab\alpha\beta} e^\alpha_c e^\beta_d < e^c \wedge e^d, e^a \wedge e^b > \epsilon \\
&= \frac{1}{2} \int R_{ab\alpha\beta} e^\alpha_c e^\beta_d \left(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}\right) \epsilon \\
&= \int R^{cd}{}_{\alpha\beta} e^\alpha_c e^\beta_d (e) d^4x
\end{aligned} \tag{201}
$$

where d^4x is a short notation for $dx^0 \wedge ... \wedge dx^{(n-1)}$.

VIII. APPENDIX 2

The purpose of this appendix is to give the physical meaning of the resonances

$$P_1 = 0 \tag{202}$$

$$P_2 = 0 \tag{203}$$

$$P_1 = P_2 \tag{204}$$

First let us write the metric:

$$q_{11} = \frac{-1}{16P_1 G} \frac{(Ku)^2 - 16\gamma^2}{Ku} \exp\left(-2\frac{\theta_2}{G}\right) \exp\left(-2P_1\frac{\theta_3}{G}\right) \tag{205}$$

$$q_{22} = \frac{-1}{16P_2 G} \frac{(Ku)^2 - 16\gamma^2}{Ku} \exp\left(-2\frac{\theta_1}{G}\right) \exp\left(-2P_2\frac{\theta_3}{G}\right) \tag{206}$$

$$q_{33} = \frac{-1}{2G} P_1 P_2 (\sqrt{P_1} + \sqrt{P_2})^2 \frac{Ku}{(Ku)^2 - 16\gamma^2} \tag{207}$$

where

$$u \equiv \exp\left(-\sqrt{P_1 P_2}\frac{\theta_3}{G}\right) \tag{208}$$

$$\gamma = \sqrt{P_1 P_2}(\sqrt{P_1} + \sqrt{P_2})^2 \tag{209}$$

We can rewrite this in a more convenient manner. To recover real general relativity, we choose P_1 and P_2 imaginary and write separately the real and imaginary part of θ_3. We finally find

$$q_{11} = \frac{-i(\sqrt{P_1} + \sqrt{P_2})^2}{2G}\sqrt{\frac{P_2}{P_1}} \cosh\left(\sqrt{P_1 P_2}\frac{\theta_3}{G}\right) \exp\left(-2\frac{\theta_2}{G}\right) \exp\left(-2P_1\frac{\theta_3}{G}\right) \tag{210}$$

$$q_{22} = \frac{-i(\sqrt{P_1} + \sqrt{P_2})^2}{2G}\sqrt{\frac{P_1}{P_2}} \cosh\left(\sqrt{P_1 P_2}\frac{\theta_3}{G}\right) \exp\left(-2\frac{\theta_1}{G}\right) \exp\left(-2P_2\frac{\theta_3}{G}\right) \tag{211}$$

$$q_{33} = \frac{1}{G}\frac{i\sqrt{P_1 P_2}}{\cosh\left(\sqrt{P_1 P_2} G^{-1}\theta_3\right)} \tag{212}$$

where we write θ_3 instead of $\mathrm{Im}[\theta_3]$ to simplify the notation. If $P_1 = P_2$, $q_{11} = q_{22}$.

Recall that, in the standard Misner notation,

$$q_{11} = \exp 2\left(-\Omega + \beta_1 + \sqrt{3}\beta_2\right) \tag{213}$$

$$q_{22} = \exp 2\left(-\Omega + \beta_1 - \sqrt{3}\beta_2\right) \tag{214}$$

Thus

$$q_{11} = q_{22} \Rightarrow \beta_2 = 0 \tag{215}$$

Viewing the evolution of the universe as a scattering process in a triangular potential well, this gives us a particle-universe running right into the right hand channel.

Let's see now the case $P_2 \to 0$. In this case

$$q_{11} \to \infty \tag{216}$$

while the other two components of the metric go to zero. This correspond to a particle going into one of the left hand channel. Similar behavior occurs with $P_1 \to 0$.

IX. APPENDIX 3

Let's go back to our exact integral for I_k :

$$I_k = \frac{1}{k!}\int_0^\infty x^k \exp(-x)\left(1 + B\exp(-x)\right)^{k/2} dx \tag{217}$$

We can write

$$(1+B\exp(-x))^{k/2} = 1 + \left[\sqrt{1+B\exp(-x)}-1\right]\sum_{i=0}^{k-1}\left(\sqrt{1+B\exp(-x)}\right)^i \tag{218}$$

If $|B|<1$, then

$$\sqrt{1+B\exp(-x)} \leq 1+\frac{1}{2}B\exp(-x) \tag{219}$$

and

$$\sum_{i=0}^{k-1}\left(\sqrt{1+B\exp(-x)}\right)^i \leq \sum_{i=0}^{k-1}\left(\sqrt{1+B}\right)^i \tag{220}$$

$$= \frac{\left(\sqrt{1+B}\right)^k-1}{\sqrt{1+B}-1} \tag{221}$$

Thus

$$\begin{aligned} I_k &= \frac{1}{k!}\left\{\int_0^\infty x^k\exp(-x)\,dx + \int_0^\infty x^k\exp(-x)\left[\sqrt{1+B\exp(-x)}-1\right]\sum_{i=0}^{k-1}\left(\sqrt{1+B\exp(-x)}\right)^i dx\right\} \\ &= 1+\frac{1}{k!}\int_0^\infty x^k\exp(-x)\left[\sqrt{1+B\exp(-x)}-1\right]\sum_{i=0}^{k-1}\left(\sqrt{1+B\exp(-x)}\right)^i dx \end{aligned} \tag{222}$$

now

$$\begin{aligned} \int_0^\infty x^k\exp(-x)\left[\sqrt{1+B\exp(-x)}-1\right]\sum_{i=0}^{k-1}\left(\sqrt{1+B\exp(-x)}\right)^i dx &\leq \frac{1}{4}B\frac{\left(\sqrt{1+B}\right)^k-1}{\sqrt{1+B}-1}\frac{1}{2^k}\int_0^\infty y^k\exp(-y)\,dy \\ &= \frac{1}{4}B\frac{\left(\sqrt{1+B}\right)^k-1}{\sqrt{1+B}-1}\frac{1}{2^k}k! \end{aligned}$$

Therefore

$$I_k = 1+F_k(B) \tag{223}$$

where

$$F_k(B) \leq \frac{1}{4}B\frac{\left(\sqrt{1+B}\right)^k-1}{\sqrt{1+B}-1}\frac{1}{2^k} \tag{224}$$

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